Dynamic uncapacitated lot sizing with random demand under a fillrate constraint

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Seminar für SCM und Produktion
Universität zu Köln

EURO Conference 2009

Bonn, July 2009
Agenda

1. Introduction
   - The Problem
   - Solution Approaches

2. Optimization Model
   - Formulation

3. Solution approaches
   - Exact solution
   - Heuristic solution

4. Numerical Results
   - Experiment 1
   - Experiment 2

5. Conclusion
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5 Conclusion
Planning situation
Dynamic and Random Demand

- Demand (forecasted averages and variations)

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- Holding costs
- Setup costs
- Service level
Alternatives

- **Common sense approach (MRP, APS)**
  Compute safety stocks and add to forecasted demand

- 
  
  
  \((s_t, q_t)\)-policy, \((r_t, S_t)\)-policy
  
  Use a **stationary** inventory policy with dynamic adjustment of parameters

- **"Static-dynamic uncertainty" strategy**
  Fix replenishment **periods** in advance, adjust production quantity

- **"Static uncertainty" strategy**
  Fix replenishment **periods** and **quantities** in advance
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Model Formulation I

Model $\text{SSIULSP}^q_{\beta_c}$:

Minimize $Z = \sum_{t=1}^{T} \left( s \cdot \gamma_t + h \cdot E \{ [l_t]^+ \} \right)$ \hspace{1cm} (1)

s.t.

$l_{t-1} + q_t - D_t = l_t \quad t = 1, 2, \ldots, T \hspace{1cm} (2)$

$q_t - M \cdot \gamma_t \leq 0 \quad t = 1, 2, \ldots, T \hspace{1cm} (3)$

$l_{t}^{f, \text{prod}} = -[l_{t-1} + q_t]^- \quad t = 1, 2, \ldots, T \hspace{1cm} (4)$

$l_{t}^{f, \text{end}} = -[l_t^-] \quad t = 1, 2, \ldots, T \hspace{1cm} (5)$

$F_t = l_{t}^{f, \text{end}} - l_{t}^{f, \text{prod}} \quad t = 1, 2, \ldots, T \hspace{1cm} (6)$
Model Formulation II

\[ l_t = (l_{t-1} + 1) \cdot (1 - \gamma_t) \quad t = 1, 2, \ldots, T \]  

\[ l_0 = -1 \]  

\[ \omega_t = \gamma_{t+1} \quad t = 1, 2, \ldots, T - 1 \]  

\[ \omega_T = 1 \]  

\[ 1 - \frac{\mathbb{E}\left\{ \sum_{j=t-l_t}^t F_j \right\}}{\mathbb{E}\left\{ \sum_{j=t-l_t}^t D_j \right\}} \geq \beta^*_c \quad t \in \{ t \mid \omega_t = 1 \} \]
Symbols used I

- $\beta^*_c$: target fillrate
- $D_t$: demand in period $t$ (random variable)
- $F_t$: backorder in period $t$ (random variable)
- $\gamma_t$: binary setup indicator in period $t$
- $h$: inventory holding cost
- $I_t$: net inventory at the end of period $t$ (random variable)
- $I_{t,\text{end}}^f$: backlog at the end of period $t$ (random variable)
- $I_{t,\text{prod}}^f$: backlog immediately after production in period $t$ (random variable)
- $l_t$: number of periods since the last setup prior to period $t$
- $M$: large number
Symbols used II

\( \omega_t \)  
indicator variable: \( \omega_t = 1 \), if production takes place in period \( t + 1 \); \( \omega_t = 0 \), otherwise

\( q_t \)  
production quantity in period \( t \)

\( s \)  
setup cost

\( T \)  
length of planning horizon

\( [x]^+ \)  
\( = \max\{0, x\} \)

\( [x]^− \)  
\( = \min\{0, x\} \)
Expected Inventory

\[
E\{I_t^p\} = \int_0^{Q(t)} (Q(t) - y) \cdot f_{Y(t)}(y) \cdot dy
\]

\[
= Q(t) - E\{Y(t)\} + G_{Y(t)}^1(Q(t))
\]

\[t = 1, 2, \ldots \quad (14)\]

\(Q(t)\) – cumulated production quantity from period 0 to \(t\)
\(Y(t)\) – cumulated demand from period 0 to \(t\)
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Shortest-Path Network

\[ E\{C_{\tau t}\} = s + h \cdot \sum_{\ell=\tau}^{t-1} E \left\{ \left[ l_{\tau-1}(P_{\tau}) + q_{\tau t}^* - \sum_{i=\tau}^{\ell} D_i \right]^+ \right\} \] (15)
## Shortest-Path Network

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<th>Setup in period</th>
<th>On hand inventory $E{I_5^p}$</th>
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**Table:** Expected on-hand inventory at the end of period 5 as a function of the setup pattern
Solution Procedure

1. $\mathcal{M} := \mathcal{U} := \{2, 3, \ldots, T\}$
2. $\mathcal{E} := \{(\tau, t) | \tau = 1, 2, \ldots, T; t = \tau + 1, \tau + 2, \ldots, T - 1\}$
3. for all $((0, t) \in \mathcal{E}$ with $t \in \mathcal{U})$ do
   4. Predecessor$(t) := 1$; $C(t) := \mathcal{E} \{C_{1t}\}$
   5. end for
6. while ($\mathcal{M} \neq \emptyset$) do
5. Select $\tau \in \mathcal{M}$ with minimum $C(\tau)$
8. $\mathcal{M} := \mathcal{M} \setminus \tau$; $\mathcal{U} := \mathcal{U} \setminus \tau$
9. if ($\tau = T$) then
   10. end
11. else
   12. for all $((\tau, t) \in \mathcal{E}$ with $t \in \mathcal{U})$ do
      13. $\mathcal{M} := \mathcal{M} \cup t$
      14. if ($C(\tau) + \mathcal{E} \{C_{\tau t}\} < C(t)$) then
         15. Predecessor$(t) := \tau$
         16. $C(t) := C(\tau) + \mathcal{E} \{C_{\tau t}\}$
         17. end if
     18. end for
   19. end if
20. end while
Dynamic Lot Sizing Heuristic

1: $\tau := 1$
2: while ($\tau < T$) do
3: \hspace{1em} $t := \tau$
4: \hspace{1em} while ($t < T$) do
5: \hspace{2em} if ($C_{\tau t} \leq C_{\tau,t+1}$) then
6: \hspace{3em} $t := t + 1$
7: \hspace{2em} else
8: \hspace{3em} Make current lotsize for period $\tau$ permanent.
9: \hspace{2em} $\tau := t + 1$
10: \hspace{1em} end if
11: \hspace{1em} end while
12: end while
Silver-Meal Rule

\[
E\{C_{\tau t}\} = \frac{s + h \cdot \sum_{\ell=\tau}^{t} E \left\{ l_{\tau-1}(P_{\tau-1}) + q_{\tau}^* - \sum_{i=\tau}^{\ell} D_i \right\}^+}{t - \tau + 1}
\] (16)
Least-Unit-Cost rule

\[ E\{C_{\tau t}\} = E \left\{ \frac{s + h \cdot \sum_{\ell=\tau}^{t} \left[ I_{\tau-1}(P_{\tau-1}) + q_{\tau t}^* - \sum_{i=\tau}^{\ell} D_i \right] +}{\sum_{i=\tau}^{t} D_i} \right\} \] (17)
Least-Total-Cost rule

\[ E\{C_{tt}\} = E\left\{ s + h \cdot \sum_{\ell=\tau}^{t} \left[ l_{\tau-1}(P_{\tau-1}) + q^*_{\tau t} - \sum_{i=\tau}^{\ell} D_i \right]^+ \right\} \quad (18) \]
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Expected Demands

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Parameters

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## Results

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<tr>
<th>Series #</th>
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Results for Demand Series 1
Parameters

\[ E\{D_t\} \sim \text{Uniform}(0, 100) \]

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Conclusion

- Exact solution for the stochastic Wagner-Whitin problem
- Adjusted cost criteria used in standard dynamic lot sizing heuristics
- Silver-Meal rule superior to Groff rule
- Directly applicable in ERP/AP systems
- Static uncertainty strategy: no nervousness, no bullwhip effect
- Target service level (instead of backorder costs)
- Possible extension: Capacities (done)